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On the Existence of States Saturating the Bogomol'nyi Bound in N=4 Supersymmetry

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ABSTRACT

We give an argument showing that in N=4 supersymmetric gauge theories there exists at least one bound state saturating the Bogomol'nyi bound with electric charge p and magnetic charge q, for each p and q relatively prime, and we comment on the uniqueness of such state. This result is a necessary condition for the existence of an exact S-duality in N=4 supersymmetric theories.

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A mounting body of evidence [1, 2, 3] supports the conjecture that N=4 rigid supersymmetric theories possess an $SL(2,Z)$ duality. This $SL(2,Z)$ contains as a subgroup the strong-weak duality of Montonen and Olive [4, 5, 6].

Let us confine our attention to an N=4 theory with gauge group $SU(2)$ spontaneously broken to $U(1)$, for simplicity. In this case the duality acts by fractional transformations on the complex number $S = \theta/2\pi + i4\pi/g^2$, which combines together the theta angle θ and the gauge coupling constant g

$$S \rightarrow \frac{aS+b}{cS+d}, \quad a,b,c,d \in Z, \quad ad-bc=1. \tag{1}$$

The meaning of duality is that theories whose coupling constants are related by a transformation of type (1) are physically equivalent. In particular, the stable physical states of the two theories are the same, up to an eventual relabeling of them. Among the physical states of the theory there is a special subset: those that saturate the Bogomol'nyi bound [7]. These states belong to short multiplets of N=4 and thus their mass is non renormalized [8]. By denoting with $p, q \in Z$ the electric and magnetic charge of one such state, its mass is given by the following formula [8, 2]

$$M^2(p,q) = \frac{4\pi V^2}{\mathrm{Im}\, S}(p^2 + 2\mathrm{Re}\, Spq + |S|^2 q^2). \tag{2}$$

Here V is a constant independent of p, q and S.

Not all assignments of p and q need give a stable state. Indeed, a (p,q) state may decay in a pair $[(p_1,q_1),(p_2,q_2)]$ $p_1+p_2=p$, $q_1+q_2=q$, whenever

$$M(p,q) \geq M(p_1,q_1) + M(p_2,q_2). \tag{3}$$

Since eq. (2) defines a positive-definite scalar product ($\mathrm{Im}\, S > 0$), the triangular inequality holds: $M(p,q) \leq M(p_1,q_1) + M(p_2,q_2)$, with the equality attained when $p_1/q_1 = p_2/q_2 = p/q$. The charges are integers, thus this is possible only when p and q have a common divisor

$$p = Nm, \quad q = Nn, \quad N,n,m \in Z. \tag{4}$$

This argument, due to Sen [2, 3], does not guarantees that states obeying the above equations actually decay, but it does shows that all Bogomol'nyi states with p, q relatively prime are stable. Moreover, Bogomol'nyi states are transformed by S-duality. If this duality exists, eq. (2) must be invariant under the $SL(2,Z)$ transformation (1), together with a relabeling of p and q, which turns out to be

$$p \rightarrow ap - bq, \quad q \rightarrow -cp + dq. \tag{5}$$

If S-duality holds, eq. (5) maps the multiplet with $p=1, q=0$ (the electrically charged vector multiplet of the $SU(2)$ theory, corresponding to a broken generator of the gauge group) into states with $p=a$, $q=-c$. These states must also be stable, by consistency. This property holds because the constraints $ad-bc=1$, $a,b,c,d \in Z$ imply that p and q are relatively prime.

Stable states with $p=0,1$ and $q=0,1$ are well known: they are, respectively, the elementary charged states of the $SU(2)$ gauge theory, the BPS monopole, and the dyon. A crucial test of S-duality is to verify whether *all* states with p,q relatively prime do indeed exist with the right

multiplicity. The case $q=2$, p odd integer, has been studied in [3]. The topological aspects of the generalization to arbitrary q have been demonstrated by Segal [9].

Purpose of this paper is to show that for all p,q relatively prime, there exists at least one N=4 (short) multiplet of states whose mass saturates the Bogomol'nyi bound (2), and to propose a possible strategy to prove its uniqueness.

In [3], a general strategy for finding Bogomol'nyi states with arbitrary electric and magnetic charge has also been proposed. Let us resume the features of that argument that we shall need in this paper.

Let us put as in [3] $\theta=0$, the general case being (probably) reconducible to this one once the Witten phenomenon [10] is taken into account. Let us also notice that it is sufficient to study our N=4 theory in the small-coupling constant (semiclassical) regime: $g\rightarrow 0$. Indeed, Bogomoln'yi states with p,q relatively prime are stable for each value of g i.e. they cannot disappear (decay) as the coupling constant is adiabatically switched off, or turned on.

In the semiclassical limit, the dynamics of states with magnetic charge q is determined as follows [11]. One looks for *static* solutions of the classical equations of motion of the N=4 $SU(2)$ theory, with magnetic charge $q>0$ (the case $q<0$ is analogous), and mass given by eq. (2) $M=(4\pi/g)Vq$. They depend on several bosonic and fermionic moduli, that we denote collectively by m. Let us call these solutions $\phi^o(\vec{x},m)$. We may then decompose the most general N=4 field (better, multiplet of fields), denoted by $\phi(t,\vec{x})$, as

$$\phi(x)\equiv\phi(t,\vec{x})=\phi^o(\vec{x},m(t))+\delta^T\phi(t,\vec{x}). \tag{6}$$

Note that the moduli have been given a time dependence. If one expands the N=4 action, $S[\phi(t,\vec{x})]$, up to quadratic terms (higher terms are irrelevant for $g\rightarrow 0$) one finds that the linear terms are absent because ϕ^o is a solution of the classical equations of motion, and the action reads

$$S[\phi(t,\vec{x})]=S[\phi^o(\vec{x},m(t))]+\int d^4xd^4y\delta^T\phi(x)\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\delta^T\phi(y) \tag{7}$$

The fluctuations $\delta^T\phi(x)$ are transverse, i.e. they obey

$$\int d^3x\frac{\delta^2 S}{\delta m(t)\delta\phi(x)}\delta^T\phi(x)=0,\quad\forall t \tag{8}$$

The quantization of the classical theory with lagrangian (7) gives a hamiltonian of the form

$$H=H^o[m,-i\partial_m]+H^T[\delta^T\phi(\vec{x}),\delta^T\pi(\vec{x}),m]. \tag{9}$$

The "transverse hamiltonian," H^T, does not depend on $-i\partial_m$ because of eq. (8), thus m enters in H^T as an external parameter. It can be shown that upon quantization the energy levels of H^T are $(4\pi/g)Vq+\mathcal{O}(1)$, and that by supersymmetry the ground state energy of H^T is independent of m and equal to $(4\pi/g)Vq$. If a static Bogomol'nyi state with electric charge p exists, instead, its energy is $(4\pi/g)Vq+(g^3/8\pi q)Vp^2+\mathcal{O}(g^5)$. Such state, therefore, is a product $\psi(m)\otimes\Psi^0$, where Ψ^0 denotes the ground state of H^T. Ψ^0 depends on m, but, as usual in the $g\rightarrow 0$ limit

(a.k.a. the Born-Oppenheimer approximation), this dependence is negligible; thus, $\psi(m)$ is a *normalizable* eigenstate of the *quantum mechanical* hamiltonian $H^0 \equiv H^o + (4\pi/g)Vq$, with eigenvalue $(4\pi/g)Vq + (g^3/8\pi q)Vp^2$.

We have just proven that to find a state with charges p, q, whose mass saturates the Bogomol'nyi bound, one must find a normalizable bound state of the hamiltonian H^0, which describes the quantum mechanics on the (finite-dimensional) q-monopole moduli space.

The hamiltonian H^0 has been studied in [12] and it turns out to be a N=4 quantum mechanical sigma model on the q-monopole moduli space.

This space is hyperkähler [13] and has the structure

$$M_q = R^3 \times \frac{S^1 \times M_q^0}{Z_q}. \tag{10}$$

Here, R^3 denotes the configuration space of the center of mass coordinate of the monopole, while S^1 is labelled by the coordinate χ, with the identification $\chi \approx \chi + 2\pi$. The metric on $R^3 \times S^1$ is flat, whereas M_q^0 is a hyperkähler *non-compact* manifold of real dimension $4(q-1)$ [13]. The total electric charge of the system is the conjugate variable to χ: $Q_{el} = -i\partial/\partial\chi$. The discrete group Z_q acts freely on S^1: $z\chi = \chi + 2\pi/q$, and also acts non-trivially on M_q^0. The quantization of the N=4 sigma model on M_q can be done, following ref. [3], by quantizing at first on the covering space

$$\tilde{M}_q = R^3 \times S^1 \times M_q^0, \tag{11}$$

and then truncating the Hilbert space by projecting on Z_q invariant wave functions. Let us write explicitly the hamiltonian H^0. To do this we must explicitly list the moduli, previously denoted collectively by m. They are [12]:

- 4 real bosonic moduli Y^a, $a = 1,..,4$ parametrizing $R^3 \times S^1$. They may be denoted equivalently by $\vec{Y}$, $\chi \equiv Y^4$.
- 2×4 fermionic moduli, η^a_α, $\alpha = 1, 2$ related by supersymmetry to the Y^a.
- $4(q-1)$ real bosonic moduli X^A, $A = 1,..,4(q-1)$, parametrizing M_q^0.
- $2 \times 4(q-1)$ real fermionic moduli, λ^A_α, related by supersymmetry to the X^A.

The hamiltonian reads

$$H^0 = H^1 + H^2, \quad H^1 = -\frac{1}{2} g^{q\,ab} \frac{\partial}{\partial Y^a} \frac{\partial}{\partial Y^b} + \frac{4\pi V}{g} q. \tag{12}$$

The metric on $R^3 \times S^1$, as stated before, is flat. Its normalization, as well as the constant term $(4\pi/g)Vq$, may be fixed by demanding that the eigenvalues of H^1 reproduce the nonrelativistic, $g \to 0$ limit of the mass formula (2):

$$g^{q\,ab} = \mathrm{diag}\left(\frac{g}{8\pi Vq}, \frac{g}{8\pi Vq}, \frac{g}{8\pi Vq}, \frac{g^3 V}{8\pi q}\right). \tag{13}$$

Here we use a real notation to describe the moduli of $R^3 \times S^1$ and M^0_q. In this notation, only one of the four supersymmetries of the model is manifest. By denoting with g_{AB}, Γ^A_{BC}, and R_{ABCD}, the metric, Christoffel symbol, and Riemann curvature of M^0_q, respectively, and setting $D_A = \partial_A - \Gamma^B_{AC} g_{BD} \bar{\lambda}^C \gamma^0 \lambda^D$ (notations are as in [12, 3]), the hamiltonian H^2 reads [12, 14]

$$H^2 = -\frac{1}{2\sqrt{g}} D_A \sqrt{g} g^{AB} D_B - \frac{1}{12} R_{ABCD} \bar{\lambda}^A \lambda^C \bar{\lambda}^B \lambda^D. \tag{14}$$

Since the hamiltonian H^0 is a sum of two commuting operators, its eigenfunctions factorize as $\psi = \psi^1 \otimes \psi^2$. ψ^1 is an eigenstate of H^1. Notice that H^1 is independent of the 8 fermionic coordinates η^a_α. Thus the eigenvalues of H^1 are 16-fold degenerate, and ψ^1 is the product of a bosonic wave function, a plane wave $\exp(iP_a Y^a)$, and a fermionic wave function ξ. The fermionic wave function is a state in the 16-dimensional fermionic Fock space. A simple calculation shows that for each value of P_a, the 16 degenerate states correspond to a 4-dimensional, massive spin-one particle, four spin-1/2 (Majorana), and five real spin-0 particles. This is exactly the content of a massive short multiplet of N=4 supersymmetry in four dimensions! Moreover,

$$H^1 \exp(iP_a Y^a)\xi = E^1 \exp(iP_a Y^a)\xi = \left(\frac{g}{8\pi V q} \vec{P}^2 + \frac{g^3 V}{8\pi q} p^2 + \frac{4\pi V}{g} q \right) \exp(iP_a Y^a)\xi. \tag{15}$$

Recalling that $\vec{P}$ is the center of mass momentum of the q-monopole, and that $p \equiv P_4$ is the electric charge, we find that indeed eq. (15) gives the non-relativistic limit of the energy of a Bogomol'nyi state. The spectrum of H^2 is positive definite, by supersymmetry: $H^2\psi^2 = E^2\psi^2$, $E^2 \geq 0$.

Since the total energy of a q-monopole configuration is $E^1 + E^2$, this implies that Bogomol'nyi states are in one-to-one correspondence with *normalizable* solutions of the equation $H^2\psi^2 = 0$, in other words, they are in one-to-one correspondence with the zero-energy eigenvalues of H^2. Notice that for a given value of q and $p \equiv P_4$, the degeneracy of the eigenvalue E^1 already gives a complete N=4 short multiplet with mass given by eq. (2). Thus, one would obtain an N=4 short multiplet saturating the Bogomol'nyi bound for each zero-energy eigenstate of H^2. Since at $p = 1, q = 0$ there is only one such multiplet, S-duality predicts that a zero energy eigenstate of H^2 with charges p, q relatively prime is non-degenerate.

So far, we have mostly recalled the arguments in ref. [3]. Now we come to the main point of the paper, namely, to prove the existence of normalizable solutions of the equation

$$H^2\psi^2 = 0. \tag{16}$$

If the bosonic moduli space M^0_q were compact, then the spectrum of H^2 would be discrete and one could count straigthforwardly the number of solutions of eq. (16). One way of doing it would be to perturb the hamiltonian H^2 by adding a N=1 superpotential $W(X)$ [15]. The superpotential is a real, smooth function of X^A, in terms of which the ordinary potential energy $V(X)$ reads $V(X) = (1/2) g^{AB} \partial_A W(X) \partial_B W(X)$. As shown in ref. [15], when the spectrum of a supersymmetric hamiltonian is discrete [2], the index $\Delta = N_B - N_F$, i.e. the number of

[2] Or when, at least, the continuous part of the spectrum is always larger than a nonzero, positive number, i.e., when the spectrum has a gap.

bosonic minus fermionic zero-energy states of a supersymmetric system, is invariant under a large class of supersymmetry-preserving deformations of the hamiltonian, which includes the addition of a superpotential, and can be computed in the semiclassical approximation. In the semiclassical approximation, the zero-energy states of the model are determined by the points at which the potential $V(X)$ vanishes, i.e. by the stationary points (denoted here by X_i) of the superpotential (where $\partial W/\partial X^A|_{X_i} = 0 \; \forall A$). If they are isolated and the hessian matrix, $H_{AB\,i} \equiv \partial^2 W(X)/\partial X^A \partial X^B|_{X_i}$, is non-degenerate, $N_B - N_F = \sum_i (-)^{\sigma_i}$. Here σ_i is the number of negative eigenvalues of the hessian $H_{AB\,i}$ [3]. Moreover, if the hamiltonian is invariant under a symmetry group (Z_q, in our case), the Hilbert space of the model decomposes into irreducible representations of that group, and one can apply the previous arguments separately to each representation. Thus, $\Delta^p = N^p_B - N^p_F$, i.e. the number of bosonic ground states in the representation p, minus the fermionic ones, is an index [15]. Unfortunately, M^0_q is noncompact and the spectrum of H^2 is gapless. Nevertheless, we can still add an F-term perturbation, i.e. a superpotential $(1/n)W$, where n is a positive integer, and then let $n \to \infty$. In this way, for each finite n, and by choosing an appropriate $W(X)$, we can still have a mass gap, and use Witten's argument to prove that there exists a normalizable zero-energy state for each stationary point of W. Obviously, we must carefully study the fate of those normalizable (bound) states as $n \to \infty$. Indeed, nothing guarantees *a priori* that a zero-energy bound state of the perturbed theory, ψ^2_n, would converge to a *normalizable* ground state of H^2.

To solve this problem, our strategy will be the following:

1. we will introduce a superpotential $W(X)$, well defined and nonsingular on the whole M^0_q, invariant under Z_q, and with q isolated zeroes such that $\Delta^p \geq 1$, for $p = 1, .., q-1$.
2. We will add the superpotential $(1/n)W(X)$ to the hamiltonian H^2, and let $n \to \infty$. We will find that the perturbed ground states ψ^2_n converge pointwise to $\mathcal{C}^\infty(M^0_q)$ solutions, ψ_∞, of the equation $H^2\psi_\infty = 0$.
3. By studying the large-distance behavior of ψ_∞ we will find that they are normalizable whenever p and q are relatively prime, *as required by S-duality.*
4. Finally, we will exhibit a superpotential obeying the properties assumed in point 1, and we will comment on the uniqueness of the BPS bound states.

Adding a superpotential $(1/n)W(X)$ to H^2 modifies the hamiltonian as follows

$$H^2_n = H^2 + \frac{1}{2n}\nabla_A \nabla_B W \bar{\lambda}^A \lambda^B + \frac{1}{2n^2} g^{AB} \nabla_A W \nabla_B W \equiv H^2 + \delta H^2_n \tag{17}$$

where ∇_A is the usual covariant derivative on M^0_q, obtained from the Christoffel connection.

The complete wave function on M_q, $\psi_n = \psi^1_n \otimes \psi^2_n$, is invariant under Z_q. Since the electric charge is $Q_{el} = -i\partial/\partial Y^4$, a state $\psi_{n\,p}$ of electric charge p is the product of two wave functions transforming as follows under Z_q:

$$z\psi^1_{n\,p} = \exp(i2\pi p/q)\psi^1_{n\,p}, \quad z\psi^2_{n\,p} = \exp(-i2\pi p/q)\psi^2_{n\,p}. \tag{18}$$

[3] In one dimension, the sign of Δ is only a matter of conventions: one may also set $N_F - N_B = \sum_i (-)^{\sigma_i}$ by redefining the Fock vacuum.

Thus, by denoting with $|0\rangle$ the Fock vacuum of the fermionic Hilbert space, defined by $\lambda^A|0\rangle = 0$ $\forall A$, one may write the most general solution of the equation $H^2_n\psi^2_{n\,p} = 0$ as

$$\psi^2_{n\,p} = \sum_{i=0}^{4(q-1)} \varphi^{\alpha_1,A_1,...\alpha_i,A_i}_{n\,p}(X^A)\prod_{j=1}^{i}\bar{\lambda}^j_{\alpha_j}|0\rangle, \quad (A_l,\alpha_l)\neq(A_m,\alpha_m), \quad \text{for } l\neq m. \tag{19}$$

By substituting formula (19) into the equation $H^2_n\psi^2_{n\,p} = 0$, one can re-write it as a system of $2^{4(q-1)}$ elliptic second-order partial differential equations. They have the following structure:

$$-\frac{1}{2}g^{AB}\partial_A\partial_B\varphi^{\alpha_1,A_1,...\alpha_i,A_i}_{n\,p} + \text{lower derivative terms} = 0. \tag{20}$$

This is a special case of elliptic system for which many of the theorems valid for one-component elliptic equations apply [16].

Now, let us study the fate of $\psi^2_{n\,p}$, as $n\to\infty$. First of all, since $\Delta^p \geq 1$, for $p = 1,..,q-1$, at finite n, there exists at least one normalizable $\psi^2_{n\,p}$ for each $p = 1,...,q-1$. Since all coefficient of the system (20) are $\mathcal{C}^\infty(M^0_q)$, the functions $\varphi^{\alpha_1,A_1,...\alpha_i,A_i}_{n\,p}$ are also $\mathcal{C}^\infty(M^0_q)$. Obviously, given an arbitrary compact set $K'\subset M^0_q$, one may normalize the wave function, uniformly in n, as follows

$$\max_{X\in K',i}\max_{\alpha_1,A_1,...\alpha_i,A_i}|\varphi^{\alpha_1,A_1,...\alpha_i,A_i}_{n\,p}(X)| = 1. \tag{21}$$

Because of this equation, given a $\mathcal{C}^0(M^0_q)$ function f, and denoting by Ω the volume of K', one has

$$|\int_{K'}\varphi^{\alpha_1,A_1,...\alpha_i,A_i}_{n\,p} f| \leq \Omega \max_{K'}|f|. \tag{22}$$

In K', the hamiltonian H^2_n converges pointwise to H^2 as $n\to\infty$ This property, together with a standard theorem in analysis, ensures that the $\varphi^{\alpha_1,A_1,...\alpha_i,A_i}_{n\,p}|_{K'}$ converge in the weak* topology to distributions $\varphi^{\alpha_1,A_1,...\alpha_i,A_i}_{\infty\,p}$. Moreover, since $H^2_n\psi^2_{n\,p} = 0$ for all n, one has $H^2\psi^2_{\infty\,p} = 0$. The coefficients of H^2 are $\mathcal{C}^\infty(M^0_q)$, thus H^2 is a hypoelliptic system of differential equations, and the $\varphi^{\alpha_1,A_1,...\alpha_i,A_i}_{\infty\,p}(X)|_{K'}$ belong indeed to $\mathcal{C}^\infty(K')$ (see for instance chapter IV of ref. [17]).

Since the previous convergence result are valid on an arbitrary compact set K', we find that the coefficients $\varphi^{\alpha_1,A_1,...\alpha_i,A_i}_{n\,p}(X)$ of a charge-p wave function ψ^2_p are the point-wise limit

$$\varphi^{\alpha_1,A_1,...\alpha_i,A_i}_{p}(X) = \lim_{n\to\infty}\sum_{k=0}^{(q-1)}\exp(i2\pi pk/q)\varphi^{\alpha_1,A_1,...\alpha_i,A_i}_{n}(z^kX). \tag{23}$$

Thus, eq. (23) proves that, for each value of $p = 1,...,q-1$, there exists at least one zero-energy wave function ψ^2_p. The coefficients $\varphi^{\alpha_1,A_1,...\alpha_i,A_i}_{p}(X)$ of this wave function are $\mathcal{C}^\infty(M^0_q)$, but they are not necessarily bounded on M^0_q. In other words, not all these ψ^2_p correspond to bound states: only the normalizable ones do; our next task is to find them.

Since the coefficients of ψ^2_p are smooth, we need only study the asymptotic behavior of them. To do this we must understand the structure of the asymptotic regions of M^0_q. This can be done as follows: the space M_q is a desingularization of the symmetric product of q one-monopole spaces $R^3\times S^1$ [13]. Its asymptotic regions correspond to configurations where

these q monopoles can be split into two well separated sets: a set of k monopoles, with center of mass $\vec{Y}_k$, and a set of $q-k$ monopoles, with center of mass $\vec{Y}_{q-k}$. In this case, since a multi-monopole configuration is localized in space [4], one can write the charge-q solution $\phi^o_q(\vec{x}, m_q(t))$ (cfr. eq. (6)) as an approximate linear superposition

$$\phi^o_q(\vec{x}, m_q(t)) \approx \phi^o_k(\vec{x}, m_k(t)) + \phi^o_{q-k}(\vec{x}, m_{q-k}(t)). \tag{24}$$

Moreover, since the action $S[\phi(t, \vec{x})]$ itself is local, one finds

$$S[\phi^o_q(\vec{x}, m_q(t))] \approx S[\phi^o_k(\vec{x}, m_k(t))] + S[\phi^o_{q-k}(\vec{x}, m_{q-k}(t))]. \tag{25}$$

This equation implies that the metric of the moduli space factorizes when $d \equiv |\vec{Y}_k - \vec{Y}_{q-k}| \to \infty$:

$$\lim_{d\to\infty} g^q_{IJ} = \mathrm{diag}(g^k_{I_1 J_1}, g^{q-k}_{I_2 J_2}), \quad I_1, J_1 = 1, ..., 4k, \;\; I_2, J_2 = 4k+1, ..., 4q. \tag{26}$$

Equivalently, asymptotically $\tilde{M}_q$ factorizes (locally) into $\tilde{M}_k \times \tilde{M}_{q-k}$. More precisely, by denoting with U^q_d a neighborhood of a point $m_q \in \tilde{M}_q$, such that $|\vec{Y}_k - \vec{Y}_{q-k}| = d$, the following local factorization property holds: $\lim_{d\to\infty} U^q_d = U^k \times U^{q-k}$. Here U^k and U^{q-k} are neighborhoods of $\tilde{M}_k$ and $\tilde{M}_{q-k}$, respectively. It should be possible to prove rigorously these properties using the results of ref. [18].

Eq. (26) implies that the hamiltonian $H^1 + H^2_n$ also factorizes asymptotically. To study it let us parametrize $\tilde{M}_k = R^3 \times S^1 \times M^0_k$ with the following coordinates: Y^a_k, parametrizing $R^3 \times S^1$, and $X^{A_1}_k$, parametrizing M^0_k. Let us similarly parametrize $\tilde{M}_{q-k}$ by Y^a_{q-k}, $X^{A_2}_{q-k}$, and define

$$\begin{aligned} \vec{Y} &= \frac{k}{q}\vec{Y}_k + \frac{q-k}{q}\vec{Y}_{q-k} \\ \vec{Z} &= \vec{Y}_k - \vec{Y}_{q-k}. \end{aligned} \tag{27}$$

The factorization of the metric given in eq. (26) then implies that, when $d \equiv |\vec{Z}| \to \infty$ the asymptotic form of the equation for $\psi_{n\,p}$ is

$$\begin{aligned} &\left(\frac{4\pi}{g} V q + \frac{g}{8\pi V q}\vec{P}^2 + \frac{g^3 V}{8\pi q}p^2\right)\psi_{n\,p}(\vec{Z}) = (H^1 + H^2_n)\psi_{n\,p}(\vec{Z}) \approx \\ &\approx \left(-\frac{1}{2}g^{k\,ab}\frac{\partial}{\partial Y^a_k}\frac{\partial}{\partial Y^b_k} - \frac{1}{2}g^{q-k\,ab}\frac{\partial}{\partial Y^a_{q-k}}\frac{\partial}{\partial Y^b_{q-k}} + \frac{4\pi V}{g}q + +H^2_k + H^2_{q-k} + \delta H^2_n\right)\psi_{n\,p}(\vec{Z}). \end{aligned} \tag{28}$$

Here we have emphasized the dependence of $\psi_{n\,p}$ on $\vec{Z}$ (i.e. the relative distance between the two multi-monopole clusters). The metrics $g^{k\,ab}$ and $g^{q-k\,ab}$ are as in eq. (13), whereas the hamiltonians H^2_k, H^2_{q-k} are as in eq. (14). Equation (28) also implies that asymptotically the wave function $\psi_{n\,p}(\vec{Z})$ is a linear superposition of vectors $\exp(iP_a Y^a)\Psi_{n\,p_1,p_2,\mu}(\vec{Z})$, where

$$\begin{aligned} (H^2_k + H^2_{q-k} + \delta H^2_n)\Psi_{n\,p_1,p_2,\mu}(\vec{Z}) &= \mu\Psi_{n\,p_1,p_2,\mu}(\vec{Z}), \\ -i\frac{\partial}{\partial Y^4_k}\Psi_{n\,p_1,p_2,\mu}(\vec{Z}) &= p_1\Psi_{n\,p_1,p_2,\mu}(\vec{Z}), \\ -i\frac{\partial}{\partial Y^4_{q-k}}\Psi_{n\,p_1,p_2,\mu}(\vec{Z}) &= p_2\Psi_{p_1,p_2,\mu}(\vec{Z}). \end{aligned} \tag{29}$$

[4] This is most easily seen in the singular gauge.

Here $p_1, p_2 \in Z$, $p_1 + p_2 = p$ and $\mu \geq 0$. This last property holds because the spectrum of the supersymmetric hamiltonian $H^2_k + H^2_{q-k}$ is non-negative and, away from the stationary points of the potential, where the semiclassical approximation holds, $\delta H^2_n \approx (1/2n^2) g^{AB} \nabla_A W \nabla_B W \geq 0$. Substituting eq. (29) into eq. (28) one finds

$$\left(-\frac{gq}{8\pi V k(q-k)} \frac{\partial}{\partial \vec{Z}} \frac{\partial}{\partial \vec{Z}} + \frac{g^3 V}{8\pi k} p_1^2 + \frac{g^3 V}{8\pi (q-k)} p_2^2 - \frac{g^3 V}{8\pi q} p^2 + \mu \right) \Psi_{n\, p_1, p_2, \mu}(\vec{Z}) = 0. \tag{30}$$

This is an equation in $\vec{Z}$ whose normalizable solutions behave asymptotically as a Yukawa potential:

$$\Psi_{n\, p_1, p_2, \mu}(\vec{Z}) \approx \mathrm{const}\, |\vec{Z}|^{-1} \exp(-M_n(p_1, p_2, k, \mu)|\vec{Z}|), \tag{31}$$

where

$$M_n(p_1, p_2, k, \mu) = V \sqrt{\frac{g^2(q-k)}{q} p_1^2 + \frac{g^2 k}{q} p_2^2 - \frac{g^2 k(q-k)}{q^2} p^2 + \frac{8\pi k(q-k)}{gqV} \mu}. \tag{32}$$

This equation gives a bound, *independent of* n, on the asymptotic behavior of the wave function $\psi_{n\, p}$:

$$\psi_{n\, p}(\vec{Z}) \leq \mathrm{const}\, |\vec{Z}|^{-1} \exp[-M(k)|\vec{Z}|], \tag{33}$$

where

$$M(k) = \min_{p_1, p_2 \in Z} M_\infty(p_1, p_2, k, 0). \tag{34}$$

Thus, when M is strictly positive, eq. (33) guarantees that the wave function $\psi_{n\, p}(\vec{Z})$ is normalizable in $\vec{Z}$, uniformly in n. If $M(k) > 0$ for any $k = 1, .., q-1$, then the wave function is *uniformly* bound by a normalizable wave function in all asymptotic regions of M^0_q. Since the bound is uniform in n, the wave function ψ_p is also normalizable.

Now, we have reduced the problem of finiding zero-energy bound states of H^2 with charge p to the problem of finding a pair of charges (p, q) such that $\min_{k=1,..,q-1} M(k) > 0$.

An inspection of eq. (32) shows that $M^2(p_1, p_2, k, 0)$ is a positive semidefinite quadratic form in p_1, p_2 which vanishes at $p_1/k = p_2/(q-k)$. As we already pointed out, this is impossible if p and q are relatively prime. Thus, whenever p and q are relatively prime, there exists at least one zero-energy bound state of H^2: this is consistent with the prediction of S-duality!

At this point, we must exhibit a superpotential $W(X)$ with the desired properties, and comment on the arbitrariness involved in the procedure so far described. The choice of $W(X)$ is far from unique: as we have just seen, the bound on the asymptotic behavior of the wave function is independent on the asymptotic form of the perturbing superpotential. Thus, to complete our proof we only need to exhibit explicitly a superpotential obeying the conditions in point 1 above. To do this, we use the representation of M^0_q given in ref [13]. There, M^0_q was represented by the set of coefficients $(a, b) \in C^{2q-1}$ obeying the algebraic constraint $\Delta(a, b) = 1$,

where $\Delta(a,b)$ is the resultant

$$\Delta(a,b) = \det \begin{pmatrix} a_0 & a_1 & \dots & \dots & a_{q-1} & & & & \\ & a_0 & \dots & \dots & \dots & a_{q-1} & & & \\ & & \dots & \dots & \dots & \dots & & & \\ & & & & a_0 & \dots & \dots & \dots & a_{q-1} \\ b_0 & b_1 & \dots & b_{q-2} & 0 & 1 & & & \\ & b_0 & \dots & \dots & b_{q-2} & 0 & 1 & & \\ & & \dots & \dots & \dots & \dots & \dots & & \\ & & & b_0 & \dots & \dots & b_{q-2} & 0 & 1 \end{pmatrix}. \tag{35}$$

We set

$$W(a,b) = \kappa \sum_{i=1}^{q-1} \left(\frac{|a_i|^2}{\sum_{l=0}^{k-1} |a_l|^2} + |b_{i-1}|^2 \right). \tag{36}$$

Here κ is an arbitrary nonzero constant [5]. This function is symmetric under Z_q and it has exactly q isolated stationary points at

$$a_i = b_{i-1} = 0, \quad i = 1,..,q-1, \quad a_0 = \exp(2\pi i r/q), \quad r = 0,..,q-1. \tag{37}$$

To prove this, we introduce a Lagrange multiplier λ which implements the constraint $\Delta(a,b) - 1 = 0$ and we write the stationarity conditions subject to the constraint.

$$\begin{aligned} \frac{\partial}{\partial \bar{a}_k}\{\lambda[\Delta(a,b)-1] + \bar{\lambda}[\bar{\Delta}(a,b)-1] + W(a,b)\} &= 0, \\ \frac{\partial}{\partial \bar{b}_k}\{\lambda[\Delta(a,b)-1] + \bar{\lambda}[\bar{\Delta}(a,b)-1] + W(a,b)\} &= 0, \\ \frac{\partial}{\partial a_k}\{\lambda[\Delta(a,b)-1] + \bar{\lambda}[\bar{\Delta}(a,b)-1] + W(a,b)\} &= 0, \\ \frac{\partial}{\partial b_k}\{\lambda[\Delta(a,b)-1] + \bar{\lambda}[\bar{\Delta}(a,b)-1] + W(a,b)\} &= 0, \\ \Delta(a,b) &= 1. \end{aligned} \tag{38}$$

Since $W(a,b)$ and $\Delta(a,b)$ are homogeneous functions in a_k, of degree 0 and q respectively, the first equation in (38), multiplied by a_k and summed over k, gives $\lambda = 0$. The second equation then implies $b_i = 0$, $i = 0, ..., q-2$. At $b_i = 0$, $\Delta(a,b) = a_0^q$, and the constraint implies $a_0^q = 1$ i.e. $|a_0| = 1$; thus, the third equation gives $a_i = 0$, $i = 1, ..., q-1$.

There are many other ways to construct superpotentials with the required properties[6]. These superpotentials break the $N = 4$ supersymmetry of the sigma model to $N = 1$. In $N = 1$ supersymmetry, we know how to control the difference $N_B^p - N_F^p$, but not N_B^p or N_F^p separately; thus, there exists the (unlikely) possibility of finding extra zero-energy normalizable states with p, q relatively prime.

[5]Notice that by choosing the constant κ large enough, we can insure that the semiclassical approximation holds to arbitrary accuracy everywhere outside of the stationary points.

[6]The superpotentials W are not completely arbitrary, though. Indeed, since the action of Z_q on M_q^0 is free [13], their stationary points always fall into orbits of order q; thus, $\Delta^p \geq 1$ for $p = 0, ..., q-1$.

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